

UNIQUE ECCENTRIC POINT GRAPHS

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A graph G is defined to be a unique eccentric point graph (a u.e.p. graph) if each point of G has a unique maximum distance point. Here u.e.p. graphs which are self-centered are characterized. Two construction procedures are given for generating families of u.e.p. graphs from known u.e.p. graphs. Furthermore, some special classes of u.e.p. graphs are studied in detail.

1. Introduction

For general notation and terminology, we follow Harary [5].

The graphs considered here are ordinary connected graphs. Let G be a graph with point set $V(G) = \{u_1, u_2, \dots, u_p\}$. We define below some distance-related concepts for G . When the graph to which the parameters relate is obvious from the context we may omit reference to the graph.

The distance $d_G(u, v)$ between points u and v is the length of the shortest u - v path in G . The eccentricity $e_G(u)$ of the point u is given by $e_G(u) = \max\{d_G(u, v) \mid v \in V(G)\}$. The radius r_G and diameter d_G are defined as follows. $r_G = \min\{e_G(u) \mid u \in V(G)\}$ and $d_G = \max\{e_G(u) \mid u \in V(G)\}$. (Note. For $u \in V(G)$, $d_G(u)$ denotes the degree of u ; that is the number of lines incident with u .) Let (e_1, e_2, \dots, e_p) be the sequence of eccentricities of the points of G arranged such that $e_1 \leq e_2 \leq \dots \leq e_p$. Then this sequence is called the eccentricity sequence of G . A point v is called a peripheral point of G if $e_G(v) = d_G$ and a central point if $e_G(v) = r_G$. The set of peripheral points of G is denoted by $P(G)$.

For $u \in V(G)$, let $N_i(u) = \{v \in V(G) \mid d(u, v) = i\}$. For $S \subseteq V(G)$, the induced subgraph on S is denoted by $\langle S \rangle$. For $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, the subgraph induced by the set of lines of G having one end in S and the other in T is denoted by $[S, T]$. A graph G is defined to be upper-diameter critical if $d_G = d$ and $d_{G+e} < d$ for every $e \in E(\bar{G})$, where \bar{G} is the complement of G .

We define a point v to be an eccentric point of u if $d_G(u, v) = e_G(u)$. In general, a point v is called an eccentric point if it is an eccentric point of some point u , and is called a non-eccentric point otherwise. Let $E(u)$ denote the set of all eccentric points of u . We define a graph G to be a unique eccentric point graph, (a u.e.p. graph for short) if $|E(u)| = 1$ for every $u \in V(G)$. The unique eccentric point of u is denoted by \bar{u} .

So far, very little work has been done on eccentricity sequences and the only known references on this seem to be the papers by Lesniak [6] and Behzad and Simpson [1]. The graphs for which $r=d$, or which have constant eccentricity sequences have been defined as self-centered graphs and were studied by Capobianca [4] and Buckley ([2] and [3]). In this paper we study the properties of u.e.p. graphs as part of the general problem of studying graphs with given eccentricity properties. We note that the u.e.p. graphs which are self-centered are referred to as diametrical graphs in Mulder [7], where two examples are given to show that not all diametrical graphs are regular or bipartite.

In Section 2, some general properties of u.e.p. graphs are studied and u.e.p. graphs which are self-centered are characterized. In the next section we give two construction procedures for generating families of u.e.p. graphs from known u.e.p. graphs. Here, we prove that the cartesian product of any two u.e.p. graphs is again a u.e.p. graph. And finally in Section 4, we characterize the trees and graphs with diameter two which are u.e.p. graphs and also prove some interesting results on u.e.p. graphs having diameter three.

2. General properties

A few examples of u.e.p. graphs are given in Fig. 1. The n -cube is a u.e.p. graph which is n -regular with diameter n . The path P_{2n} and the cycle C_{2n} are u.e.p. graphs.

It is well known that for any graph G , $r_G \leq d_G \leq 2r_G$. The first proposition shows that the upper bound is not attained for a u.e.p. graph.

Proposition 1. *For every u.e.p. graph $d \leq 2r - 1$.*

Proof. The proof follows from the fact that all peripheral points of G are at distance r from any central point when $d = 2r$.

Proposition 2. *In any u.e.p. graph G , $|P(G)|$ is even.*

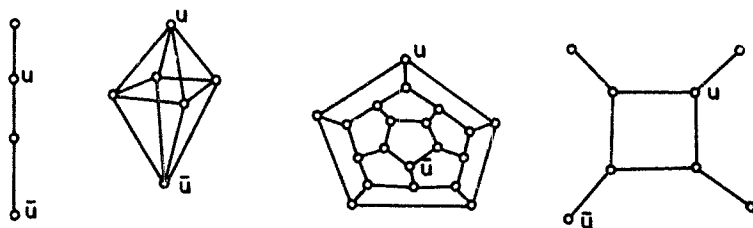


Fig. 1.

Proof. The proof follows from the fact that for any peripheral point x , we have $\bar{x} = x$.

Proposition 3. *If xy is a line of a u.e.p. graph and $e(x) \neq e(y)$, then $\bar{x} = \bar{y}$.*

Proof. Let xy be a line of G with $e(x) \neq e(y)$. Without loss of generality, assume $e(x) < e(y)$. For every $v \in N_i(x)$, $i \leq e(x) - 1$, we have $d(y, v) \leq d(y, x) + d(x, v) \leq 1 + e(x) - 1 = e(x)$. Since $V(G) - \bigcup_{i \leq e(x)-1} N_i(x) = \{\bar{x}\}$ and $e(y) > e(x)$, it follows that $\bar{x} = \bar{y}$.

Proposition 4. *If G is a self-centered u.e.p. graph, then*

$$|N_{d-1}(v)| \geq |N_1(v)| \quad \text{for every } v \in V(G).$$

Proof. Consider $v, \bar{v} \in V(G) = P(G)$. Let $x \in N_1(v)$. Obviously, $\bar{x} \in N_{d-1}(v)$ or $\bar{x} \in N_d(v)$, the latter holding if and only if $\bar{x} = \bar{v}$. Also $d(x, \bar{v}) \geq d - 1$. If $d(x, \bar{v}) = d$, then $E(\bar{v})$ contains both x and v , a contradiction. Hence $d(x, \bar{v}) = d - 1$, so that $\bar{v} \neq \bar{x}$, and therefore $\bar{x} \in N_{d-1}(v)$. Also for $x \neq y$, $\bar{x} \neq \bar{y}$, as all points are peripheral points in G . Hence $|N_{d-1}(v)| \geq |N_1(v)|$ for every $v \in V(G)$.

Since any peripheral point of a graph is an eccentric point, it is clear that every point of a self-centered graph is an eccentric point. That the converse also is true for a u.e.p. graph is the content of the following theorem.

Theorem 1. *A u.e.p. graph G is self-centered if and only if each point of G is an eccentric point.*

Proof. By the preceding remarks we have only to prove the converse implication. Let G be a u.e.p. graph such that all its points are eccentric points. We first prove that $\bar{v} = v$ for every $v \in V(G)$. Suppose, to the contrary, that $\bar{v} \neq v$ for some point v . Choose v such that $\bar{v} \neq v$ and v has the least eccentricity among such points. As v is an eccentric point, there exists $x \in V(G)$ such that $\bar{x} = v$. Note that $e(x) \leq e(v)$.

Case 1: $e(x) = e(v)$. Then $\bar{v} = x$ and hence $\bar{v} = \bar{x} = v$, a contradiction.

Case 2: $e(x) < e(v)$. Then $\bar{x} = \bar{v} \neq x$ as $e(v) > e(x)$. This contradicts the choice of v .

Hence $\bar{v} = v$ for every $v \in V(G)$. Thus

$$e(\bar{v}) = e(v) \quad \text{for every } v \in V(G). \quad (1)$$

Suppose, now, that $r \neq d$. Then there exists a line xy such that $e(x) \neq e(y)$. By Proposition 3, we have $\bar{x} = \bar{y}$ ($= v$, say). Then by (1), we have $e(v) = e(\bar{x}) = e(x)$ and also $e(v) = e(\bar{y}) = e(y)$, which implies $e(x) = e(y)$, a contradiction. Hence $r = d$ and G is self-centered.

Corollary. A u.e.p. graph G is self-centered if and only if $\bar{u} = u$ for every $u \in V(G)$.

3. General constructions

Construction 1. Let G be any u.e.p. graph on p points with diameter d . Let $V(G) = \{v_1, v_2, \dots, v_p\}$. Consider p graphs G_i , $i = 1, 2, \dots, p$ all having the same diameter s and satisfying the condition that for each G_i , there exist at least two peripheral points v' and \bar{v}' such that $E(v') = \{\bar{v}'\}$. Now identify the points v_i of G with v' of G_i , $i = 1, 2, \dots, p$. The resulting graph is again a u.e.p. graph with diameter $2s + d$, which contains G as an induced subgraph.

Construction 2. The following theorem gives a construction procedure to generate u.e.p. blocks from a given u.e.p. graph or a collection of u.e.p. graphs.

Theorem 2. The cartesian product $G_1 \times G_2$ of two u.e.p. graphs G_1 and G_2 is again a u.e.p. graph.

Proof. The proof hinges on the following simple observation. For any two points $s = (u_i, v_j)$ and $t = (u_k, v_m)$ of $G_1 \times G_2$, we have

$$d_{G_1 \times G_2}(s, t) = d_{G_1}(u_i, u_k) + d_{G_2}(v_j, v_m).$$

If \bar{u}_i and \bar{v}_j are the unique eccentric points of u_i in G_1 and v_j in G_2 , respectively, then (\bar{u}_i, \bar{v}_j) is the unique eccentric point of (u_i, v_j) in $G_1 \times G_2$.

Remark. If we define a graph G to be an m -eccentric point graph (an m -e.p. graph, for short) if $|E(v)| = m$ for every $v \in V(G)$, then the following generalization of Theorem 2 holds.

Theorem. The cartesian product $G_1 \times G_2$ of an m -e.p. graph G_1 and an n -e.p. graph G_2 is an mn -e.p. graph.

4. Special classes of graphs

4.1. Trees

It is well known that for every tree $d = 2r$ or $d = 2r - 1$. In the latter case the tree is called *bicentral*.

Theorem 3. A tree T is a u.e.p. graph if and only if it is bicentral and has exactly two peripheral points.

Proof. If T is a u.e.p. tree, then by Proposition 1, we have $d \neq 2r$. Hence $d = 2r - 1$ and T is bicentral. If T has more than two peripheral points, then for at least one of the central points there exists more than one point at distance r from it, contradicting the assumption that T is a u.e.p. graph.

The validity of the converse is easily checked; hence the theorem.

4.2. U.e.p. graphs of diameter two

We note that K_2 is the only u.e.p. graph with diameter one. The next result characterizes u.e.p. graphs with diameter two.

Theorem 4. *The u.e.p. graphs of diameter two are precisely the $(p-2)$ -regular graphs on p points.*

Proof. Any u.e.p. graph G of diameter two is self-centered, since by Proposition 1, $r \neq 1$. Now since G is a u.e.p. graph, any point $x \in V(G)$ is adjacent to every point except \bar{x} as $e(x) = 2$. Hence G is $(p-2)$ -regular.

The converse is easily verified.

4.3. U.e.p. graphs of diameter three

Theorem 5. *If G is a u.e.p. graph with diameter three, then G is either a self-centered graph or an upper-diameter critical graph.*

Proof. Let G be a u.e.p. graph with diameter three. Let $P = P(G)$. If $V(G) - P = \emptyset$, then there is nothing to prove. If not, let $x \in V(G) - P$. Then $e(x) = 2$, since $r \geq 2$, by Proposition 1. Also $N_2(x) = \{\bar{x}\}$ and hence $d(x) = p - 2$.

If $|P| > 2$, it follows by Proposition 2 that $|P| \geq 4$ and hence there is a pair of peripheral points u, \bar{u} in $N_1(x)$. But then $d(u, \bar{u}) \leq 2$, contradicting the fact that $u, \bar{u} \in P$.

If $|P| = 2$, say with $P = \{u, \bar{u}\}$, then all points $x \in V(G) - P$ have degree $(p-2)$. This implies that both $\langle N_1(u) \rangle$ and $\langle N_2(u) \rangle$ are complete and $[N_1(u), N_2(u)]$ is complete bipartite. Also $N_2(u) = N_1(\bar{u})$. Hence G is upper-diameter critical. (See Theorem 2.1 on page 77 of Ore [8].)

The fact that every u.e.p. graph of diameter two is self-centered and $(p-2)$ -regular prompts us to investigate whether self-centered u.e.p. graphs with diameter three are regular. The graph of Fig. 2(a) shows that this is not true. However, such graphs have a spanning $\frac{1}{2}(p-2)$ -regular supergraph which is again a self-centered u.e.p. graph with diameter three. This is the content of the next theorem.

Theorem 6. *Any self-centered u.e.p. graph with diameter three is a spanning subgraph of a self-centered u.e.p. graph with diameter three which is $\frac{1}{2}(p-2)$ -regular.*

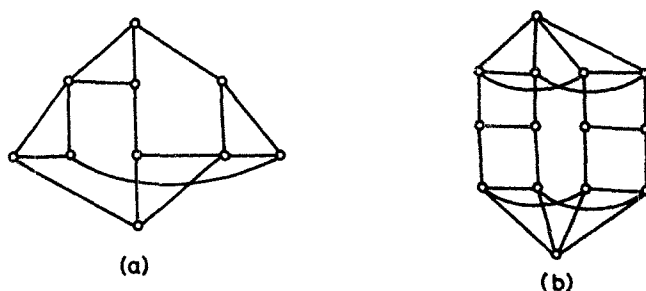


Fig. 2.

Proof. Let G be a self-centered u.e.p. graph with diameter three. It can be established using Proposition 4 that $d(v) \leq \frac{1}{2}(p-2)$ for each $v \in V(G)$. Consider $x \in V(G)$. Let $S = \{\bar{u} \mid u \in N_1(x)\}$. Then $S \subseteq N_2(x)$ and we can partition $N_2(x) - S$ into two sets T_1 and T_2 with $|T_1| = |T_2|$ such that for each $v \in T_1$, we have $\bar{v} \in T_2$ and $N_1(\bar{x}) - S \subseteq T_1$. Now join x to all points of T_2 . Then $d(x) = \frac{1}{2}(p-2)$ and it is easy to prove that this addition of new lines does not affect the u.e.p. property or the diameter of G . By adding lines for each $x \in V(G)$, we arrive at the required $\frac{1}{2}(p-2)$ -regular graph.

The result of Theorem 6 cannot be extended for self-centered u.e.p. graphs of higher diameter as shown by the graph of diameter four given in Fig. 2(b).

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